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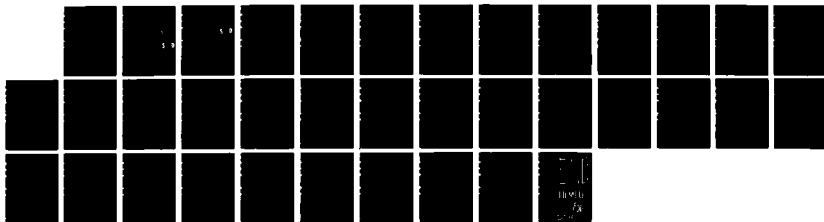
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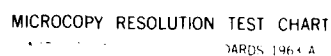
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CONTINUOUS REPRESENTATIONS OF DIGITAL
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ABSTRACT

A 2D digital image S is represented conventionally by the union of grid squares containing pixels of S which we denote by $F(S)$. This gives the correct topology for S with 8-adjacency, and with a little imagination, 4-adjacency can also be properly handled. However, one encounters difficulty in extending basic 2D results to 3D digital images. The last few years have seen the need for better methods which give a closer link with well developed continuous topology, especially with the advent of digital surface theory [11]. We define a new continuous model $\tilde{F}(S)$ by refining $F(S)$. We show that this gives a better bridge between the two subjects, digital and continuous topologies. We also show how this space $\tilde{F}(S)$ is related to two other continuous models [4] [7]. Although we concentrate only on 2D images in this paper, the concepts and general ideas extend to 3D images. A 3D version of this paper is in preparation.

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1. Introduction

A digital image is a discrete sample of a continuous picture in real life. The sample is usually taken on an evenly spaced grid point set on a rectangular domain of a plane. To be definite, think of the set of points with integral coordinates in the x - y plane as the grid point set.

For example, suppose the sample looks like the figure below:

```

1 1 0 0
0 0 1 1

```

Figure 1.1. The 1's indicate pixels of an object, and the 0's indicate pixels of the background.

The natural question is "Does this digital sample suggest one (connected) object or two separate ones of similar appearance?" Of course, it also depends on how the sample was taken, but this is not our concern here. We are interested in defining geometric concepts on digital images consistently and realistically, which is about all we can do without knowing the original picture. Returning to the connectedness question, if we regard the set of 1's shown above as a connected set, then the set of 0's should be a disconnected set in order to be consistent with the standard topology on the Euclidean plane. Thus it is customary in digital topology to use different adjacency topologies on a digital image (denoted by S) and on its background (regarded as the complementary set, and denoted by \bar{S}). For example, if 8-adjacency is used on S , which makes the 1's in Figure 1.1 connected, then 4-adjacency is used on \bar{S} which makes the 0's disconnected.



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In order to approximate the unknown original picture by a continuous one, it has been the tradition to represent each point p in the digital image S by the closed unit square around p , and to represent S by the union of all such squares. Let us denote this union by $F(S)$. This gives a good topological representation for the 8-adjacency topology on S in the sense that S is connected in this topology if and only if $F(S)$ is connected in the usual Euclidean topology. Of course, we cannot say the same thing about the 4-adjacency topology at the same time. However, it is not difficult to describe the connectedness property using generally accepted terms of Euclidean topology in $F(S)$.

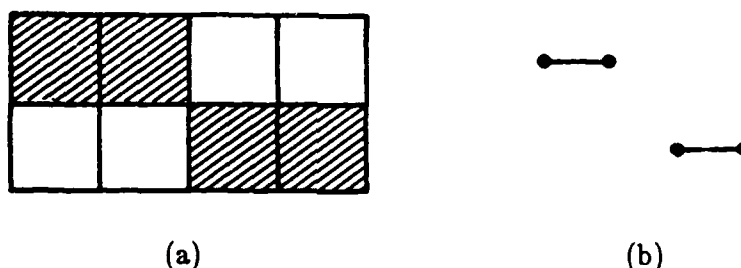


Figure 1.2. (a) $F(S)$, the traditional pixel representation. (b) $E(S)$, the skeletal representation used for computing the Euler characteristic number of S .

An alternative to the square representation has existed all along, dating back to Minsky and Papert's 1969 *Perceptrons* [8]. In computing the Euler characteristic number of a digital image S , they took a point for each point in S , a line segment for each pair of 4-adjacent points, and a square for each 2×2 matrix of four 4-adjacent points in S . This gives a good skeletal representation of S with 4-adjacency which will be denoted by $E(S)$. Of course, it is not difficult to deal with 8-adjacency as well. Curiously enough, all subsequent authors (except Mor-

genthaler [10]) used this representation only in computing Euler characteristic numbers until Kong and Roscoe used it for other purposes in 1985 [7].

As long as one is concerned with only 2D images, there is no real difficulty in defining continuous representations of digital images. The situation is different when we consider 3D images. A number of basic facts about 2D images were generalized to 3D images without great difficulty, leading to the initial development of 3D digital topology by Rosenfeld in 1980 [15], but one crucial theorem for this foundation did not yield to an easy generalization. It is Proposition 5 in [15]. The stumbling block was the fact that the corresponding theorem for 2D images used a boundary curve tracking algorithm which did not extend to surface tracking in 3D in any obvious way. Meanwhile Herman and his colleagues were developing such a surface tracking algorithm. Based on topological results from the classical book by Newman [12], Herman and Webster proved [4] that their previously developed algorithm [1] [2] was correct. However, the surface traced by their algorithm is not a surface in the strict topological sense. Nevertheless, the main result of this work was essential to the proof of Proposition 5 given by Morgenthaler and Rosenfeld in [11].

During the same period of time, a theory of digital surfaces was developed [11]. A surface was defined as a digital image satisfying a certain set of axioms analogous to the axioms for a manifold in topology. Although this definition is clever and simple, it was no easy task to prove the digital analog of the standard fact in topology that a closed surface in Euclidean 3D space is orientable [14]. Thus it was natural to seek for some closer connection between the two subjects.

the newly developed 3D digital topology and the well established Euclidean topology, so that one could use known facts from this latter discipline. Recently Kong and Roscoe did precisely that in 1985 [7]. They defined the continuous analog of a digital image as the 3D version of the representation considered by Minsky and Papert and others as mentioned above, and proved that a digital image S is a surface in the sense of Morgenthaler and Rosenfeld [11] if and only if its continuous analog is a surface in the strict topological sense. In other words, the axiomatic definition of a digital surface given by [11] was correct (if direct adjacency is used on the digital image, and indirect adjacency is used on the background image).

The purpose of this paper is to propose another method of describing a digital image S by a continuous representation to be denoted by $\tilde{F}(S)$. This representation $\tilde{F}(S)$ seems more consistent with the topological behavior of Euclidean space in the sense that it tends to give easier access to basic results from that well developed subject. We first give a new proof of the main theorem of Herman and Webster [4] from a conceptual point of view. Their proof is a combinatorial analysis of different types of strong connectedness. We reduce all these to the usual connectedness in topology using $\tilde{F}(S)$. We use this as a motivation for defining $\tilde{F}(S)$, but the advantages of $\tilde{F}(S)$ should be apparent once one sees the definition. To define it, we introduce a concept of singularity similar to that used in algebraic geometry and differential topology. We then adopt one of several possible ways of removing the singularities from $F(S)$. The resulting topological space $\tilde{F}(S)$ is a compact manifold with a boundary in the

strict topological sense. In particular, the boundary is a true surface in the 3D case, and a true curve in the 2D case. This surface is an approximation to the surface traced by the surface tracking algorithm mentioned above.

We begin with brief descriptions of $F(S)$ and $U(S)$, and with the introduction of cellular complexes in a very restricted sense (Section 2). In the following two sections, we define a 2D analog of the surface tracking algorithm by means of a graph [1] [2] [4], and we give a definition of $\tilde{F}(S)$ while constructing a proof of the 2D analog of the theorem due to Herman and Webster [4]. Combining the results in Sections 2 and 4, we give a proof of Proposition 5 [11] (Section 5). Finally, the three continuous representations of a digital image are shown to be intimately related (Section 6). For example, the 2D analog of the above mentioned surface tracking algorithm can be used to trace the continuous analog $E(S)$.

The content of this paper is devoted to 2D images, but it is so arranged as to yield an easy extension to 3D images in principle.

2. Continuous representations

We assume that the reader is familiar with the basic definitions and elementary facts about 2D digital geometry given in [16]. Let Σ be the set of all integral points within a large rectangular domain in R^2 , and let S be a subset of Σ . For each p in Σ , define $F(p)$ to be the unit closed square with p at its center, that is the set of all points (x, y) in R^2 such that $|x - p_1| \leq 1, |y - p_2| \leq 1$. Define $F(S)$ to be the union of $F(p)$ for all p in S , and $U(S)$ to be the interior of $F(S)$. (In analogy to $F(p)$, one could define the open set $U(p)$ for p in S to be the interior of $F(S)$. However, this would be quite useless since the union of the $U(p)$ for p in S consists of disjoint open sets.) Notice that $F(S)$ (resp. $U(S)$) is connected in the sense of the standard Euclidean topology if and only if S is 8- (resp. 4-) connected in the sense of digital topology. In other words, we can say that:

The 8- (resp. 4-) adjacency topology on a 2D digital image S is well represented by the standard Euclidean topology on $F(S)$ (resp. $U(S)$).

The topological spaces $F(S)$ and $U(S)$ will be referred to as *continuous representations of S (in the primitive form)*. Σ and hence S being a finite set, $F(S)$ (resp. $U(S)$) is a bounded and closed (resp. open) subset of R^2 . Thus if S is represented by $F(S)$ then the complementary set \bar{S} (called the background) should be represented by $U(\bar{S})$. If we assume that S does not meet the boundary of Σ , we have

$$U(\Sigma) = F(S) \cup U(\bar{S}).$$

This is the basic reason for using the 8-adjacency topology on S if the 4-adjacency topology is used on \bar{S} .

Now we study global properties of the topological space $F(S)$ by means of the adjacency relations among the constituent blocks $F(p)$ for p in S . The set together with the set of these blocks is a special case of a cellular complex(*) which we shall not define precisely here. One relevant remark is that it is sometimes important to distinguish the single set $F(S)$ from the collection of all its constituent cells: all squares $F(p)$ for p in S , all their edges, and all their vertices. If we denote this collection by K then we denote the union of its constituent cells by $|K|$ which is $F(S)$. The figure below shows an example,

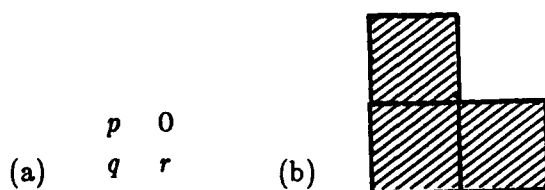


Figure 2.1. (a) $S = \{p, q, r\}$. (b) $F(S)$.

where K consists of 3 squares, 10 edges, and 8 vertices. Fortunately the cell complex K made out of the cells in $F(S)$ is uniquely determined by the single set $F(S)$. A cellular 1-complex(*) is a collection of edges in some $F(S)$ together with all their vertices. If L is a cellular 1-complex then $|L|$ denotes the union of all edges in L . For example, let $E = F(p) \cap F(q)$, the common edge of two squares, and let u, v be the two end points of E . Then $L = \{E, u, v\}$ is a cellular 1-complex. For any pixel p in Σ , define the (cellular) boundary $\partial F(p)$ to be the 1-complex consisting of its four edges and the four corner vertex points.

Notice that the set-theoretic boundary $Bd F(p)$ in R^2 is simply $|\partial F(p)|$. If $E = F(p) \cap F(q)$ with p in S , q in \bar{S} then we denote $p = i(E)$, $q = o(E)$ (i for inside, o for outside). Of course, all these depend on S . We define the (cellular) boundary $\partial F(S)$ to be the 1-complex consisting of all edges E and the end point vertices of E for which there are pixels p in S , q in \bar{S} with $E = F(p) \cap F(q)$. We leave to the reader the proof of the following important but not difficult lemma:

Lemma $|\partial F(S)| = Bd F(S)$.

(*)The cellular complex, also known as the *CW* complex, was initially defined by the English mathematician J.H.C. Whitehead in the 1930's. It consists of very general types of cells. It was invented for studying the deeper structure of a topological space, such as multiple-connectedness. It is used for computing global properties. For example, the Euler characteristic number of a *CW* complex can be computed by counting the number of cells in each dimension which is shown to depend only on the underlying topological space independent of any particular cellular complex on the space. See Spanier (1966) [17] for a more comprehensive exposition on this subject. Of course, we need only consider the very special case arising from digital topology. Klette formulated such a special type of cellular m -complex in Euclidean n -space. (See Klette (1983) [6].) It would be useful to establish, in this setting suitable to digital topology, elementary facts analogous to those in algebraic topology. The only work involving homology groups in digital topology appears in Janos and Rosenfeld (1981) [5]. This work too touches very lightly upon an algebraic method from topology.

3. Graph representation

In this section we shall define a 2D version of the graph due to Herman et al. [1] [2] [4] as a data structure for a boundary curve. Following their notation, let $Q = F(S)$ be a cellular complex, and let $P = R^2 - Q$. Two edges E, F in ∂Q are said to be P -adjacent if (1) E, F are adjacent, i.e., E, F have a common vertex, and (2) $o(E) = o(F)$, or $o(E), o(F)$ are 4-adjacent, or $o(E), o(F)$ are 4-adjacent to a common pixel in S . Notice that the first condition determines a 2×2 matrix containing all of the pixels $o(E), i(E), o(F), i(F)$, and that the second condition is equivalent to the statement that $(\mathcal{Z}) o(E), o(F)$ are 4-connected within the 2×2 matrix. There are six possible cases of relative positions of $o(E), i(E), o(F)$, and $i(F)$ within the 2×2 matrix:

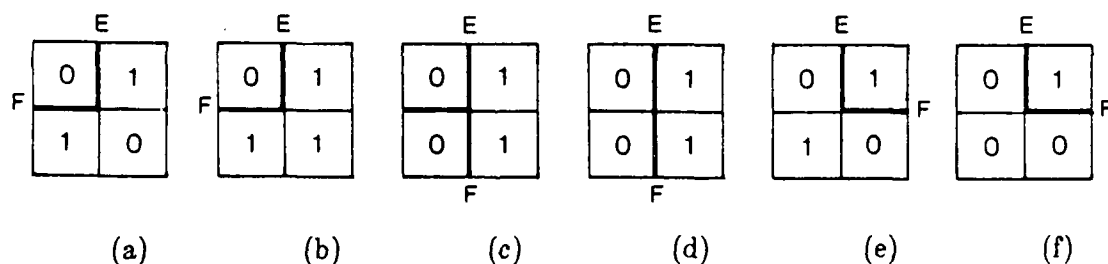


Figure 3.1.

The edges E, F in Figures 3.1 (a), (b), (c), (f) are P -adjacent whereas those in the other two figures are not. A P -component of ∂Q is defined as usual. Define a graph for ∂Q by: For each edge in ∂Q , create a node $n(E)$, and for each P -adjacent pair of edges E, F , connect the nodes $n(E), n(F)$ by an arc. We shall denote the graph by $\gamma(Q)$. We can turn $\gamma(Q)$ into a digraph by: For each p in Σ , orient $F(p)$ clockwise, and let the induced orientation on the four edges be as

indicated below:

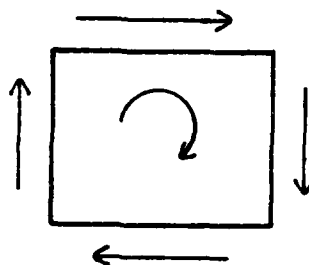


Figure 3.2.

Thus if $E = F(p) \cap F(q)$ then the two orientations on E induced by those on $F(p)$, $F(q)$ are opposite. For each edge E in ∂Q , we shall use the orientation on E induced by $o(E)$. Thus E is a directed line segment, written $E = \overrightarrow{uv}$. Define an ordered pair E, F to be **P**-adjacent if the end point of the directed line segment of E is the beginning point of the directed line segment F , that is, if $E = \overrightarrow{uv}$ then $F = \overrightarrow{vw}$. Clearly **P**-adjacency implies **P**-adjacency. A quick survey of all six cases in Figure 3.1 shows that if E, F are **P**-adjacent then either E, F , or F, E are **P**-adjacent. This then makes it possible to direct every arc of the graph $\gamma(Q)$. We shall denote the resulting digraph by $\Gamma(Q)$. The examples below reveal some interesting facts about these graphs.

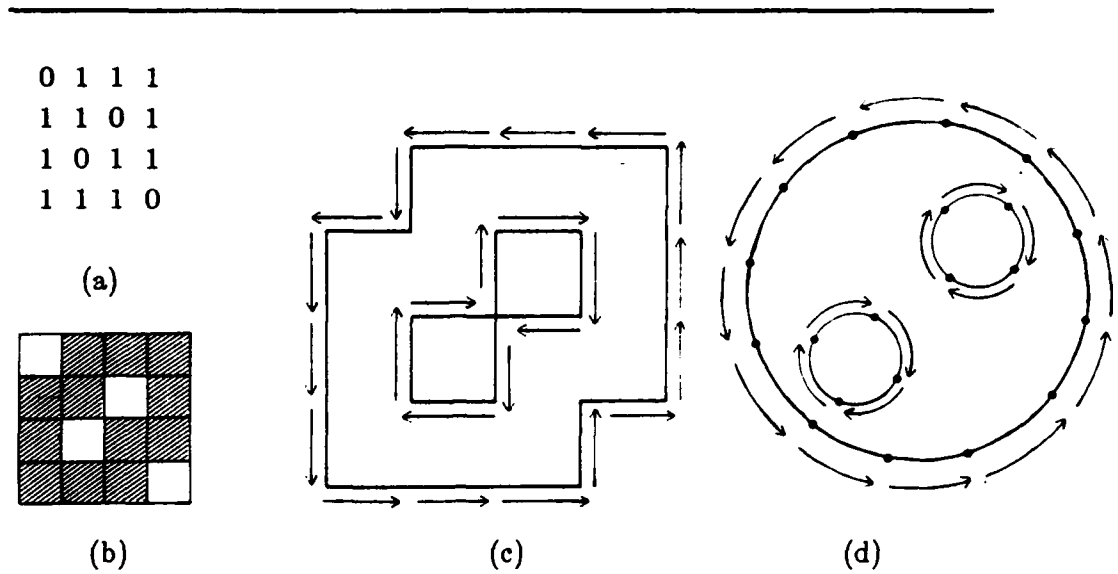


Figure 3.3. (a) S . (b) $Q = F(S)$. (c) ∂Q with directed edges. (d) Digraph $\Gamma(Q)$.

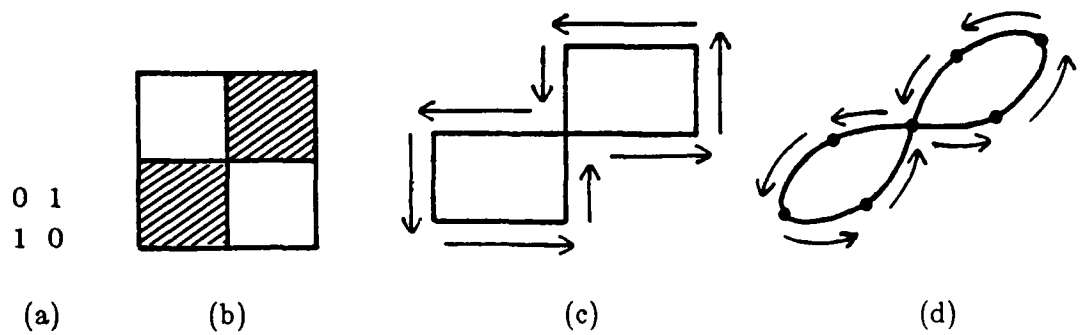


Figure 3.4. (a) S . (b) $Q = F(S)$. (c) ∂Q with directed edges. (d) $\Gamma(Q)$.

Notice that in Figure 3, $\Gamma(Q)$ has three components whereas ∂Q has only two, and that in Figure 4, both ∂Q and $\Gamma(Q)$ have only one component each.

4. Continuous representations in the refined form

The purpose of this section is to introduce a concept of singularity in a digital image S , and to show how to remove the corresponding singular point from the topological space $F(S)$ to construct a better continuous model for S . We shall do this while deducing a 2D version of the main theorem in Herman et al. [4] from a purely topological theorem proved in Appendix A.


Let $Q = F(S)$ and $P = R^2 - Q$ as before. Let C be a component of Q , and let D be a component of $R^2 - C$.

Proposition 4.1. *Bd D is a P -component of ∂Q .*

In general, a P -component is not a component of ∂Q as seen from the example in Figure 3.3c, and a P -component is not a simple closed curve as seen from the example in Figure 3.4c. It is clear from the definition that P -components are connected sets.

We need only prove that (1) every pair of points of $Bd D$ are in the same P -component; (2) if $F \subset \partial Q$ is an edge in the same P -component of an edge E in $Bd D$ then $F \subset Bd D$. First notice that $Bd D$ is a cellular 1-complex in which each edge E has the property that $U(o(E)) \subset D$, $F(i(E)) \subset C$. For (2) we need only prove it when F is actually P -adjacent to E , but then $o(E)$, $o(F)$ are in the same 4-component in $R^2 - C$ and $i(E)$, $i(F)$ are 8-connected in Q , and hence $F(i(F)) \subset C$, $U(o(F)) \subset D$ implying $F \subset Bd D$ as claimed in (2).

For (1), it follows from Proposition A in Appendix A that $Bd D$ is connected. Thus if we know that components and P -components are the same then we are done. A study of all six cases of adjacent pairs of edges E, F in ∂Q in Figure 2 showed that non- P -adjacency occurred only in a 2×2 square of the form $\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$. Such a square (or its rotation by 90 degrees) shall be referred to as a *singularity* in S . The non-existence of a singularity in S guarantees that any adjacent edges in ∂Q are P -adjacent. Thus if we assume that there is no singularity in S then we are done.

Now assume the existence of singularities in S . If $\begin{smallmatrix} 0 & p \\ q & 0 \end{smallmatrix}$ is a singularity then the common vertex point $F(p) \cap F(q)$ will be called a singular point in $F(S)$. The key problem in representing S by $F(S)$ arises from the presence of singular points: $F(S)$ looks like  near a singular point x and is connected via a single point. At the same time, the background set near x consists of two disjoint open sets but they are so close that their boundaries meet though only at a single point. This problem can be solved by modifying $F(S)$ slightly near each singularity of S . Incidentally, this kind of problem exists in many other fields such as algebraic geometry, differential topology, etc. There are several possible ways of resolving singularities, but here we shall adopt the simplest one which may be achieved in practice by taking a sufficiently fine grid. The basic idea is to widen the bottle-neck at x from $F(p)$ to $F(q)$ without altering the global picture of $F(S)$. For each singular point x , simply add to $F(S)$ a $1/4$ -size closed square σ_x with its center at x .

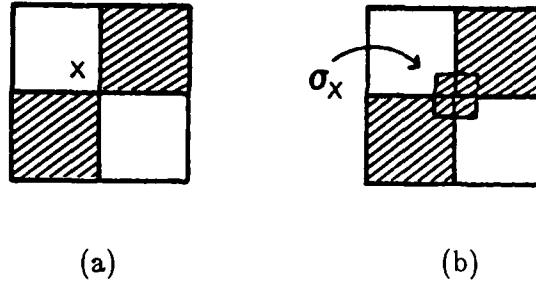


Figure 4.1. (a) Near a singular point x in $F(S)$. (b) The singular point x resolved in $\tilde{F}(S)$ by adding a small square around x .

Let $\tilde{F}(S) = F(S) \cup (\bigcup_i \sigma_{x_i})$ where x_i ranges over all singular points of $F(S)$. One could use a unit size square for σ_x , but this would alter the shape of $F(S)$ too much. Now $\tilde{F}(S)$ has no singular points. Write $\tilde{Q} = Q \cup (\bigcup_i \sigma_{x_i})$, $\tilde{C} = C \cup (\bigcup_j \sigma_{x_j})$ with x_j in C , $\tilde{D} = D - \bigcup_j \sigma_{x_j}$. Then \tilde{C} is a component in \tilde{Q} and \tilde{D} is a component in $R^2 - \tilde{Q}$. In this case, $Bd D$ has just been proved to be a component of $\partial \tilde{Q}$. Hence $Bd D$ is a simple closed polygon, and the corresponding $Bd D$ is a P -component of ∂Q . //

The space $\tilde{F}(S)$ gives more accurate topological descriptions of the digital image S , and its border sets with respect to 8-adjacency on S . The alteration of $F(S)$ altered $U(\bar{S})$ as well. Define $\tilde{U}(\bar{S})$ to be $U(\Sigma) - \tilde{F}(S)$. We shall refer to $\tilde{F}(S)$, $\tilde{U}(\bar{S})$ as *continuous representations of S with 8-adjacency (in the refined form)*.

We can define $\tilde{F}(S)$ analogously for 4-adjacency on S ; in this case the open square around x is removed from $F(S)$ for each singular point x in $F(S)$. Thus $\tilde{F}(S)$ depends on the type of adjacency used on S . When there is possibility of confusion, we shall use $F_x(S)$ for x -adjacency.

Morgenthaler considered the problem of finding conditions on S under which the topological space $F(S)$ correctly represents the digital topology on S . The conditions he proposed were phrased in terms of local connectedness. It turns out that those conditions were devised to guarantee the non-existence of a singularity in S . We shall prove this in Appendix B.

Before closing this section, we wish to point out that resolution of singularity can be achieved on the digital level. First subdivide the grid space into half-units on each axis. Thus each unit square is subdivided into 4 smaller squares. Replace each pixel by 4 new pixels of the same kind. So far, there is no real change; a singularity remains a singularity. Now we replace each singularity in S as in the figure below:

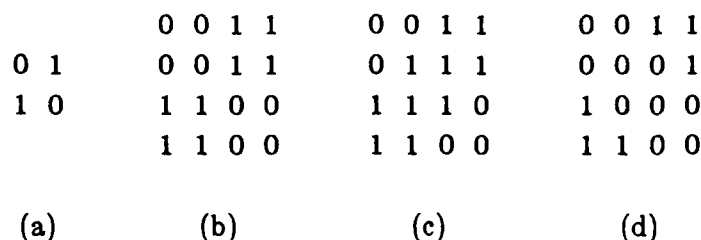


Figure 4.2. (a) A singularity. (b) Finer grid. (c) Singularity resolved for 8-adjacency on S . (d) Same for 4-adjacency on S .

It may be wiser to use a finer grid for a closer geometric approximation. We actually used a $1/4$ unit length size grid in the definition of $\tilde{F}(S)$.

5. Application to a connectedness theorem in digital topology

The following theorem on 2D images was proved in Rosenfeld and Kak [16] and its 3D analog was proved indirectly using the surface tracking algorithm due to Herman et al. in [1] [2] [4]. The purpose of this section is to show how the method of continuous representation of the previous section may be used in deducing this basic theorem in digital topology from the topological result of the previous section.

Proposition 5.1. *Let S be a digital image. Let C be a 4-component of S , and D an 8-component of \bar{S} such that C, D are 4-adjacent. Then C_D is 4-connected where C_D is the set of p in C which are 8-adjacent to D .*

Let us first prove the following exact digital analog of the theorem of the previous section:

Lemma 5.1. *Let S be a digital image. Let C be an 8-component of S , and let D be a 4-component of \bar{C} . Then D_C is 4-connected where D_C is the set of all p in D which are 8-adjacent to C .*

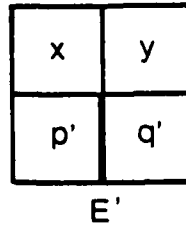
Let $Q = F(S)$. Then $F(C)$ is a component of $F(S)$. Assume that S does not meet the border of Σ and that $U(\Sigma)$ has no boundary so that $F(S) \subset U(\Sigma)$. Take any pair p, q in D_C . We want to show that p, q are in the same 4-component. By assumption, p is 8-adjacent to C . Assume p is not 4-adjacent to C . Then there is a pixel a in C located relative to p forming a 2×2 matrix

$$\begin{matrix} p & x \\ x & a \end{matrix}$$

Here the other two pixels (x 's) must be in \bar{C} . They are 4-adjacent to p , and hence are in the same 4-component as p , namely D . Let b be one of these

x 's. Then the edge $E = F(a) \cap F(b)$ is in $Bd D$. Similarly there are pixels c in C , d in D such that c, d are 4-adjacent and q, d are also 4-adjacent. The edge $G = F(c) \cap F(d)$ is in $Bd D$. Now by Proposition 4.1, E, G are in the same P -component. This implies that b, d are in the same 4-component in D_C , and hence p, q are also. //

To prove the proposition, let D' be the unique 8-component of \bar{C} containing D . In view of the above lemma, we need only show $C_D = C_{D'}$. The relation " \subset " follows immediately. Observe that C_D is the set of pixels p in C such that $F(p) \cap \partial F(D) \neq \emptyset$. Similarly for $C_{D'}$. This reduces to proving $\partial F(D') \subset \partial F(D)$. Remember that $\partial F(D)$ is the union of all edges E such that $E = F(p) \cap F(q)$ for p in D , q in \bar{D} . Similarly $\partial F(D')$ is the union of all edges E' such that $E' = F(p') \cap F(q')$ for p' in D' , q' in \bar{D}' . Actually q' is in C in the above expression of E' . If $q' \notin C$ then $q' \in D'$ since p', q' are 4-adjacent, and D' is an 8-component of \bar{C} . We need only prove that if E, E' are adjacent edges such that $E \subset \partial F(D)$ and $E' \subset \partial F(D')$ then $E' \subset \partial F(D)$. Write $E' = F(p') \cap F(q')$ as explained above. Now we need only prove that $p' \in D$ and $q' \in \bar{D}$. Since $C \cap D = \emptyset$, $q' \in C$ implies $q' \in \bar{D}$. To prove $q' \in D$, consider all possible cases of E, E' : If $p = p'$ then $p' \in D$ since $p \in D$. Suppose x or y is p in the figure below:



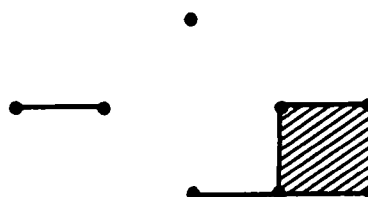
If $p' \in S$ then $p' \in C$ since p', q' are 4-adjacent, which is absurd. Thus we must have $p' \in \bar{S}$, but then $p' \in D$ since p' is 8-adjacent to p . Finally notice that $p = q'$ is not possible since $p \in D \subset \bar{S}$ and $q' \in C \subset S$. //

6. Relations among various continuous representations

In this section, we shall describe how closely the three different continuous representations of a digital surface are related. For a 2D digital image S with 4-adjacency, define a cellular complex $E(S)$ whose vertices are the mid-points $v(p)$ of the squares $F(p)$ for p in S , whose edges are the line segments connecting 4-adjacent pairs in S , and whose squares are the convex hulls of the four vertices of 2×2 matrices of 4-adjacent pixels. Notice that this is slightly more general than the definition given in Section 3. The example below should clear up any ambiguity about this definition.

0 0 1 0 0
1 1 0 1 1
0 0 1 1 1

(a)



(b)

Figure 6.1. (a) S . (b) $E(S)$.

One can easily extend this definition to higher dimensions with various types of adjacencies. Kong and Roscoe did an extensive study of 3D digital topology in [7], in which they prove that a digital image S with 6-adjacency is a digital surface in the sense of Morgenthaler and Rosenfeld [11] if and only if the 3D analog $E(S)$ is a surface in the strict topological sense. We shall prove in essence a 2D analog of the theorem: If a 3D digital image S is a digital surface in the sense of [11], then $F(S)$ is topologically equivalent to the cross product $E(S) \times [0,1]$ ($E(S)$ fattened), to be denoted by $\mathbf{E}(S)$ where $[0,1]$ is the closed unit interval.

Moreover, the surface tracking algorithm can be used to trace any boundary component of $\partial F(S)$, and hence any component of $E(S)$. For any pixel p , denote by $N(p)$ the set of all 8-neighbors excluding p . Recall that a pixel p in S is a 4-curve point if the following 2D analog of a simple surface point is satisfied: $N(p) \cap S$ has exactly two components 4-adjacent to p , $N(p) \cap \bar{S}$ has exactly two components 4-adjacent to p , and every q in $N(p) \cap S$ is 8-adjacent to both of these components. S is a digital 4-curve if every pixel of S is a 4-curve point. This defines a digital analog of a compact 1-manifold without boundary in the sense of continuous topology. Such a space is known to be a finite family of simple closed curves. A compact connected 1-manifold with non-empty boundary is known to be an arc. Therefore a compact 1-manifold is a finite union of a disjoint family of arcs and simple closed curves. For a 2D digital image, the following equivalent definition will be more convenient: A simple closed curve is a sequence of pixels p_1, p_2, \dots, p_n ($n > 4$) such that p_i is 4-adjacent to p_j if and only if $i = j + 1 \pmod{n}$. The restriction $n > 4$ is to exclude the two degenerate cases: S consisting of a single pixel, and S consisting of a single 2×2 matrix $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$. A digital 4-arc is defined similarly.

The following lemma is a 2D version of the analogous theorem due to Kong and Roscoe [7].

Lemma 6.1. *A digital image S is a simple closed 4-curve (resp. 4-arc) if and only if $E(S)$ is a simple closed polygon (resp. arc).*

Proof. Let S be a simple closed 4-curve. Using the second definition above, it is clear that $E(S)$ does not contain an isolated point or a square, and that if S consists of p_1, p_2, \dots, p_n then $E(S)$ consists of vertices $v(p_i)$ and line segments joining only every consecutive pair $v(p_i), v(p_j)$ for $j = i + 1 \pmod{n}$. In other words, $E(S)$ is a simple closed polygon.

Conversely let $E(S)$ be a simple closed polygon with vertices v_1, v_2, \dots, v_n labeled in consecutive order. Let p_i be the unique pixel in S such that $v(p_i) = v_i$. Since an isolated single point is not regarded as a simple closed curve, S cannot contain just one pixel. Similarly S cannot be a single 2×2 matrix like $\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}$.

Hence we must have $n > 4$. Now it is clear that the sequence p_1, p_2, \dots, p_n satisfies the condition for a simple closed curve. (The proof for the statement in the parentheses is similar.) //

The following theorem gives a partial characterization of a simple closed 4-curve, and at the same time, it reveals a close relation between the two continuous representations of a digital image.

Proposition 6.2. *Let S be a 2D digital image. If S is a simple closed 4-curve then $\tilde{F}(S) \cong E(S)$ (homeomorphism).*

Proof. Suppose that S is a simple closed 4-curve. If S contains no singularity then $\tilde{F}(S) \cong F(S)$, and $F(S)$ is clearly homeomorphic with $E(S)$ as claimed, and we are done. We study how a singularity may occur in S . Pick any pixel p in S . By definition, there are exactly two other pixels 4-adjacent to

p . There are only two possible cases (up to an equivalent transformation) as in the figure below:

$$\begin{array}{ccc} x & x & x \\ 1 & p & 1 \\ x & x & x \end{array} \quad \begin{array}{ccc} x & x & x \\ 1 & p & x \\ x & 1 & x \end{array}$$

(a) (b)

Figure 6.2.

No singularity can occur in Figure 6.2(a). A singularity may occur in Figure 6.2(b), but only in the upper right 2×2 sub-matrix. This shows that $F(S)$ is definitely not a product with $[0,1]$ but that $\tilde{F}(S)$ is as claimed. //

The converse is not true. Let S be a digital set of two 4-adjacent pixels. Then $\tilde{F}(S) \cong F(S)$, and certainly $F(S) \cong E(S)$, but S is not a simple closed 4-curve. Notice that S is an arc. Indeed we have

Proposition 6.3. *If $\tilde{F}(S) \cong E(S)$ (homeomorphism) then S is a disjoint union of 4-arcs and simple closed 4-curves, and conversely.*

Proof. Assume $\tilde{F}(S) \cong E(S)$. There are one-one correspondences between the following sets: components of $E(S)$, components of $F(S)$, 4-components of S , components of $\tilde{F}(S)$. The conclusion follows from Lemma 6.1.

The converse follows easily. //

So far, a digital analog of a surface with boundary has not been defined in the literature. With a suitable definition, one should be able to prove the following conjecture: *A 3D image S is a surface with boundary if and only if*

$\tilde{F}(S) \cong E(S)$ (homeomorphism).

Remark 6.4. The 2D analog of the surface tracking algorithm of Herman et al. traces $\partial F(S)$.

A rigorous proof is left to the interested reader. We shall merely study the generic examples given in Figures 3.3 and 3.4. Let S be the digital image with 4-adjacency in Figure 3.3a which is a simple closed 4-curve. The $\tilde{F}(S)$ is a fat-tened simple closed polygon $E(S)$. The suggested tracking algorithm can be used for S with 8-adjacency, and it traces out the boundary curves of 4-components of the complement of S , that is just S . In our example, there are two separate boundary curves. The interior curve looks like Figure 3.4c, which is topologically equivalent to its graph in Figure 3.4d. It is an important fact that *the curves traced by the above mentioned tracking algorithm are not in general simple closed polygons.*

Next let S be the same image, but with 8-adjacency. Then there are three components in $U(S)$, two of which are holes and the third of which is outside S . The previously stated algorithm can be applied to $Q = F(S)$. It traces out three P -components of ∂Q . In our example, they look like Figure 3.3c with the two interior squares separated from each other. Topologically, they look more like the graph in Figure 3.3d which is just $\partial \tilde{F}(S)$. (Caution: The space $\tilde{F}(S)$ depends on the adjacency used on S .) //

In this sense, we can say that *the algorithm of Herman et al. traces $E(S)$.*

7. Concluding remarks

We defined the closed subspace $F(S)$ and the open subspace $U(S)$ of Euclidean space R^2 as primitive approximation of a 2D digital image S by continuous topological spaces. This space $F(S)$ is not new. In fact, it has always been thought of as the set of pixels representing S . We pointed out that $F(S)$ (resp. $U(S)$) is a correct model for S itself with 8- (resp. 4-) adjacency, but not for its borders. However, one can get around this difficulty if one works with 2D images only, but not if we want to develop 3D digital geometry as witnessed by the difficulty encountered in the proof of orientability of a (closed) digital surface (with 6-adjacency), and by the fact that no direct proof of Proposition 5 in [10] has appeared in the literature up to now (*). The existing proof [11] depends on the continuous model of surfaces defined by Herman and his colleagues [1] [2] [4]. Thus it is clear that a good continuous model is needed so as to enable us to make use of the wealth of results from continuous topology. In this paper, we refined the model $F(S)$ to give a more accurate description of S and its borders. We first defined the concept of singular points in $F(S)$, and showed a procedure to resolve them to obtain the refined model denoted by $\tilde{F}(S)$. We produced a new proof of the main theorem of Herman and Webster [4] using $\tilde{F}(S)$ by first

(*) We believe that the reason for this is that digital images are defined on a simply connected space. All existing proofs of Proposition 5 depend on the theorem of Herman and Webster whose proof is based on topological results on polyhedral sets in Euclidean space, which is more than simply connected. In a private conversation with T.Y. Kong, we agreed that no digital analog of Euclidean space has ever been effectively used so far. Therefore, if anybody wishes to pursue digital geometry from a purely logical standpoint, it will be necessary to further develop the digital analog of Euclidean geometry.

constructing a purely topological analog, and then examining the differences between the digital and the analogous topological statements. We gave another proof of Proposition 5, but still depending on the result of Herman and Webster [4].

Traditionally, there has existed another continuous model of S which we denoted by $E(S)$. It is roughly $U(S)$ thinned down. Strangely, $E(S)$ has always been used only for computing the Euler characteristic number of S , until the appearance of Kong and Roscoe [7] in 1985. (See Minsky and Papert, *Perceptrons* (1969) [8], and Rosenfeld and Kak (1980) [16].) In [7], it is proved that a 3D digital image S is a surface with 6-adjacency in the sense of Morgenthaler and Rosenfeld [11] if and only if the 3D analog of $E(S)$ is a surface (without boundary) in the sense of continuous topology.

We showed that if S is 2D analog of a digital surface in the sense of [11] then $F(S)$ is topologically equivalent to $E(S)$ (the cross product of $E(S)$ and the unit interval). The converse is in general not true, but if we modify the statement by replacing "surface" by "surface with boundary" then it may be true. As a corollary, the 2D analog of the surface tracking algorithm [1] [2] may be used to trace $E(S)$ in principle.

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APPENDIX A

The following is a topological analog of the main theorem of Herman and Webster [4].

Proposition A. *Let K be a connected cellular complex in R^n , and let D be a component of $R^n - K$. Then the boundary set $Bd(D)$ is connected.*

(A "cellular complex" here can be thought of as a finite union of cubical cells as defined in Section 2, or more generally as a CW -complex as defined in Spanier [17].)

We need a formula from algebraic topology involving the Betti numbers. For a topological space Y , a sequence of numbers called Betti numbers $b_i(Y)$ are defined for $i = 0, 1, \dots$. $b_0(Y)$ is the number of (path) components of Y , and $b_1(Y)$ is the number of independent loops in Y . For example, $b_1(Y) = 2$ for Y a torus with one hole, and $b_1(Y) = 0$ for $Y = R^n$. We need the following formula: $b_0(A \cap B) \leq b_1(A \cup B) + b_0(A) + b_0(B) - 1$, where A, B are both open subspaces of R^n with $A \cap B \neq \emptyset$, or A, B are both subcomplexes of a cellular complex with $A \cap B \neq \emptyset$. This formula can be derived from the Mayer-Vietoris exact homology sequence for the pair (A, B) . (See Spanier [17] for details.)

Proof. It follows easily that $R^n - K$ has finitely many components, one of which is D . Denote the other components by D_i ($i = 1, 2, \dots$). Then $R^n - K = D \cup (\bigcup_i D_i)$. Notice that each D_i is open in R^n since $R^n - K$ is open and R^n is locally connected. Let $K^* = K \cup (\bigcup_i D_i)$. Then

$D = R^n - K^*$, and $K^* = R^n - D$ is a closed set of R^n . D being open in R^n , $D \cap \text{Bd } D = \emptyset$. Hence $\text{Bd } D \subset \text{Bd } K$. Similarly $\text{Bd } D_i \subset \text{Bd } K$ for all i .

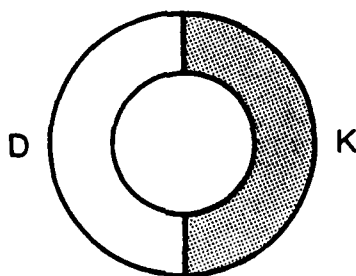
Claim: K^* is connected. It suffices to prove that if A is a non-empty subset of K^* which is open and closed in K^* then $A = K^*$. Notice that A must be closed in R^n since K^* is. K being connected, either $A \cap K = \emptyset$ or $K \subset A$. Case 1: Suppose $A \cap K = \emptyset$. Then $A \cap D_i \neq \emptyset$ for some i . D_i being connected, $D_i \subset A$. On the other hand, A being closed in R^n , $\text{Cl}(D_i) \subset A$ and hence $\text{Bd } D_i = \text{Cl}(D_i) - D_i \subset A$, but $\text{Bd } D_i \subset K$, contradicting the assumption $A \cap K = \emptyset$. Case 2: Suppose $K \subset A$. If $D_i \cap A \neq \emptyset$ then $D_i \subset A$ as observed before. Hence $A = K \cup (\bigcup_j D_j)$ where j ranges over a subset of the i 's. Suppose $D_i \cap A = \emptyset$. Let $x \in \text{Bd } D_i \subset K$. Since $\text{Cl}(D_i) \subset K^*$ and $K \subset A$ and A is open in K^* , A is an open neighborhood of x in K^* . This then means $A \cap D_i \neq \emptyset$ contradicting the assumption $D_i \cap A = \emptyset$. Hence A must contain all D_i 's, which means $A = K^*$. This proves the claim.

Now we apply the formula for $A = \text{Cl}(D)$, $B = K^*$. Notice $\text{Bd } D = A \cap B$ and $R^n = A \cup B$. Since $b_0 \text{Cl}(D) = b_0(K^*) = 1$, and $b_0(R^n) = 0$, we conclude that $b_0(\text{Bd } D) = 1$. //

This proposition is not in the most general form, which is of no interest to us. The important question here is "What makes the topological form so easy whereas the purely digital form seems so difficult?" The critical difference seems

to be the lack of efficient use of the fact that digital images are taken in Euclidean space. The above proof uses the property of simple connectedness of Euclidean space. Perhaps that is all we need from Euclidean space. If we replace R^2 by a non-simply connected space, the proposition may not be true. We end this section with a brief explanation of this phenomenon.

A space Y is said to be simply connected if Y is (path-wise) connected, and every loop in Y is deformable to a point. For example, an open n -ball and an n -sphere in R^{n+1} are simply connected, but a simple closed curve is not. The above theorem is not true if we replace "simply connected" by the weaker condition "connected". For example, take an annulus for X as in the figure below, and K for the right half shown by the shaded area. Then D is the left half in the figure, and the set $Bd D$ consists of two disjoint (vertical) line segments which contradicts the conclusion of the theorem. This is because X is not simply connected.



APPENDIX B

Morgenthaler postulated conditions on S under which S is a correct sample of $F(S)$ [9]. It turns out that his conditions are equivalent to non-existence of singularities in S . Denote by $N'(p)$ the set of all 8-neighbors of p including p , by $O(S)$ the number of components in the sense of S , and by $O(X)$ the number of components of a topological space X . His conditions are:

$$M1. O(F(S) \cap N(p)) = O(S \cap N(p)), \text{ all } p \text{ in } S.$$

$$M2. O(F(\bar{S}) \cap N(p)) = O(\bar{S} \cap N(p)), \text{ all } p \text{ in } S.$$

This gives us an opportunity to characterize singularity from a different perspective.

Lemma. *The following conditions on a 2D digital image S are equivalent to each other:*

S1. S has no singularity.

S2. If p, q in S are 8-adjacent to each other then there is r in S which is 4-adjacent to both p and q .

S3. Analog of S2 for \bar{S} .

Proof. If S2 is false then there exists a 2×2 matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (or its rotation by 90 degrees) which is a singularity. This proves that S1 implies S2. By symmetry, S1 implies S3. If S1 is false then S contains a singularity which is simply a 2×2 matrix such as that above which certainly violates both S2 and S3. Thus each of S2 and S3 implies S1. //

Proposition. *Condition (M1 and M2) is equivalent to non-existence of a singularity in S .*

Proof. Recall that $O(F(S \cap N'(p)))$ equals the number of 8-components of $S \cap N'(p)$. Thus if 8-adjacency is used on S then M1 holds automatically, and M2 is easily seen to be equivalent to S1 as 4-adjacency is supposedly used on S . By symmetry, if 4-adjacency is used on S , then M1 is equivalent to S2. //

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| <p>A 2D digital image S is represented conventionally by the union of grid squares containing pixels of S which we denote by $F(S)$. This gives the correct topology for S with 8-adjacency, and with a little imagination, 4-adjacency can also be properly handled. However, one encounters difficulty in extending basic 2D results to 3D digital images. The last few years have seen the need for better methods which give a closer link with well developed continuous topology, especially with the advent of digital surface theory [11]. We define a new continuous model $F(S)$ by refining $F(S)$. We show that this gives a better bridge between the two subjects, digital and continuous topologies. We also show how this space $F(S)$ is related to two other continuous models [4] [7]. Although we concentrate only on 2D images in this paper, the concepts and general ideas extend to 3D images. A 3D version of this paper is in preparation.</p> | | | |
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